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Codimension-three bundle singularities in F-Theory

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Abstract: We study new nonperturbative phenomena in $N = 1$ heterotic string vacua corresponding to pointlike bundle singularities in codimension three. These degenerations result in new four-dimensional infrared physics characterized by light solitonic states whose origin is explained in the dual F-theory model. We also show that such phenomena appear generically in $E_7 \to E_6$ higgsing and describe in detail the corresponding bundle transition.

Keywords: F-Theory, String Duality.

Contents

1. Introduction and overview

A remarkable achievement of string theory in recent years consists of understanding various nonperturbative effects associated to the breakdown of worldsheet conformal field theory. An example which has received much attention in the literature is the small instanton singularity in heterotic string theories [1]. These singularities occur in the context of heterotic string theories compactified on a K3 surface and are associated to the simplest pointlike degenerations of the background gauge bundles. Such degenerations have been shown to result in nonperturbative effects in six dimensions which can be understood either in terms of D-brane physics [1]–[5], or more generally, from the point of view of F-theory [6]–[9]. Various nonperturbative aspects of four dimensional heterotic strings have also been studied in detail [10]–[19]. The common feature of all these effects is that they can be ultimately related to the six dimensional small instanton singularity by an adiabatic argument. They have been accordingly interpreted in terms of heterotic fivebranes wrapping holomorphic curves in the Calabi-Yau threefold. From a mathematical point of view, such CFT singularities correspond to codimension-two bundle degenerations, precisely as in the six dimensional situation.

In this paper we consider a new class of nonperturbative effects specific to $N = 1$ heterotic compactifications on Calabi-Yau threefolds. The singularities treated in the present work are qualitatively new, being associated to *codimension-three bundle degenerations.*

This is a novel class of degenerations which have not been studied in physics so far and which are specific to four dimensional $N = 1$ compactifications. As such, we expect qualitatively new infrared effects in four dimensional $N = 1$ theories which will be discussed below.

The main tool for analyzing these singularities is heterotic/F-theory duality which encodes the bundle data in the geometry of a singular Calabi-Yau fourfold. This gives a pure geometric interpretation of the perturbative heterotic spectrum and determines at the same time the nonperturbative massless spectrum associated to CFT singularities. Generically, pointlike bundle singularities are expected to result in a certain type of spacetime defect where the nonperturbative degrees of freedom are localized. While this is also the case here, the nature of the resulting defect is very hard to understand. This is caused by our poor understanding of codimension-three degenerations of solutions to the Donaldson-Uhlenbeck-Yau equation. In particular, no explicit throat-like supergravity solution is known in this case.

In order to gain some insight into the nature of these singularities, it may be helpful to highlight the most important physical aspects by comparison with the small instanton transition. The E_8 bundle acquires pointlike degenerations which can be regarded as three dimensional defects filling space-time. However, there are no such stable excitations in the bulk M-theory, therefore such a defect is effectively stuck to the nine dimensional wall. This fact makes its physical properties quite obscure since it is not clear how to identify the light states governing the dynamics.

The F-theory picture is however more explicit. As expected, the bundle singularities correspond to special points on the F-theory base where the elliptic fibration develops certain non-generic singularities. These are superficially similar to the singularities occurring in the F-theory presentation of E_8 small instantons. So one might think by analogy that each such defect would correspond to a blowup of the three complex dimensional base. In fact, this is not the case since it will be shown in section 2 that the smooth fourfold obtained by blowing up the base is not Calabi-Yau. This is in good agreement with the absence of a 'Coulomb branch' noted previously (since the size of the exceptional divisor would be related to a displacement of the defect in the M-theory bulk, which is forbidden).

Quite remarkably, it turns out that in the present case, there exist Calabi-Yau resolutions involving only fiber blowups. Recall that the resolution of the typical ADE singular fibers occurring in F-theory consists of a chain of two-spheres with specific intersection numbers in agreement with the corresponding Dynkin diagram. On top of each point in the base we have generically such a collection of spheres. A careful analysis reveals the fact that above the special singular points the resolved fiber contains an entire *complex surface,* i.e. a manifold of dimension four rather than a collection of two-spheres. Even more surprising is the fact that the occurrence of this surface is basically automatic; no extra blowups are necessary and there are no extra generators of the Kähler cone.

This has interesting consequences for physics, which are easier to understand by compactifying the four dimensional F-theory model down to three dimensions on a circle of radius R. According to standard duality, this is equivalent to M-theory on the resolved fourfold, the size of the elliptic fiber being proportional to $1/R$. The presence of the surface

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in the fiber results in new light degrees of freedom in the low energy spectrum. We can have a string corresponding to the M fivebrane wrapped on the surface S and a tower of particle states arising by wrapping membranes on holomorphic curves in S. We regard the nonperturbative massless excitations as a sign of a singularity in the heterotic $(0, 2)$ CFT. However, at the present stage it is very hard to get more insight into the low energy dynamics.

After this outline of the physics, let us describe next the precise context in which such singularities may be encountered. It is a common fact in string theory that singularities of various sorts are associated to phase transition between string vacua. As discussed in more detail later, it turns out that the pointlike singularities considered here appear generically in the context of the Higgs phenomenon in F-theory. More explicitly, we consider a typical $E_6 \to E_7$ transition corresponding to a family of singular Calabi-Yau fourfolds with generic fiber singularity E_6 which is enhanced to E_7 along a subspace of the moduli space. Technically, such an enhancement is realized by setting to zero certain parameters of the Weierstrass model. When this apparently simple transition is studied in detail one notices the presence of extra codimension-three singularities and the nonperturbative phenomena described above.

We can get a new perspective on this transition by making use of the spectral cover construction of Friedman, Morgan, and Witten [20]. At generic points in the moduli space we have a smooth holomorphic bundle of rank three. At the transition point, this bundle degenerates in a controlled manner to a singular object which is technically a coherent sheaf. Coherent sheaves have made their appearance in a number of places in physics. For example, in the linear sigma model approach to $(0, 2)$ modes [21], the monad construction of the gauge bundle often results in a reflexive coherent sheaf [22] rather than a bundle. However, at least in the examples studied in [22], reflexive sheaves define non-singular $(0, 2)$ CFT's without the exotic phenomena described above. Given the fact that singularities in string theory tend to have a universal local behavior, it is reasonable to assume that this is the generic behavior.

In fact, this is consistent with the spectral cover description of the $E_6 \to E_7$ transition. We will show in section 3 that, at the transition point, the SU(3) bundle degenerates to a non-reflexive rank three sheaf. Moreover, this sheaf admits a natural local decomposition as a sum of a rank two reflexive sheaf and the ideal sheaf of a point. After the transition, the heterotic vacuum will be described accordingly as having two distinct sectors. We have a perturbative $(0, 2)$ CFT part corresponding to the reflexive rank two sheaf, which gives an E_7 gauge group and a certain number of matter multiplets. The second sector consists of nonperturbative degrees of freedom localized at certain points in the Calabi-Yau threefold, and corresponds to the ideal sheaves. This is quite similar to the small instanton effects in six dimensions, one of the main differences being the absence of a Coulomb branch.

This concludes our brief overview of pointlike bundle singularities in $N = 1$ string theories. More details and explicit constructions are presented in the next sections. We discuss the $E_6 \to E_7$ transition in F-theory, and explain the occurrence of the surfaces S in section 2. The heterotic picture, based on the spectral cover approach, is presented in section 3. Some technical details are postponed to an appendix.

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2. F-theory

2.1 Generalities

Our starting point for the F-theory description is a Calabi-Yau fourfold which is dual to the heterotic string on a Calabi-Yau threefold with a certain $SU(3) \times E_8$ gauge bundle. The unbroken E_6 gauge group then appears as a singularity in the elliptic fiber of the Calabi-Yau fourfold.

We will choose the heterotic Calabi-Yau threefold to be elliptically fibered over a base B, which we choose to be the ruled surface $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$; this threefold has Hodge numbers $h^{1,1} = 3$, $h^{2,1} = 243$. The dual Calabi-Yau fourfold X is then elliptically fibered over a threefold base B' which can be viewed as the total space of the projective bundle $\mathbf{P}(\mathcal{O}_B \oplus \mathcal{T})$, where $\mathcal{T} = \mathcal{O}_B(-\Gamma)$, and Γ is some effective divisor in the base B, related to the class η describing the heterotic bundle by the relation $\eta = 6c_1 - \Gamma$ [23, 13, 14]. Similar models have been considered in a different context in [24].

The fourfold $\pi' : X \to B'$ is described in the vicinity of a section $S_0 \simeq B$ by the Weierstrass equation

$$y^2 = x^3 + fx + g \tag{2.1}$$

within the bundle $\mathcal{T} \oplus (\mathcal{T}^2 \otimes \mathcal{L}^2) \oplus (\mathcal{T}^3 \otimes \mathcal{L}^3)$. We have $\mathcal{L} = -K_B$ and $\mathcal{T} \simeq N_{S_0/B'}$, the normal bundle of B in B', and we denote by s one of its sections, i.e. $s \in H^0(B; N_{S_0/B'})$. The geometry of the split IV^* singularity over the section $S_0 \simeq B$ (which corresponds to E_6 gauge group) [25] is then encoded in the following expressions for f, g and the discriminant δ:

$$\begin{aligned}
f &= s^3(f_3 + sf_4) = s^3 u\,, \\
g &= s^4(q_2^2 + sg_5 + s^2 g_6) = s^4 \widetilde{v}\,, \\
\delta &= s^8\left[4s(f_3 + sf_4)^3 + 27(q_2^2 + sg_5 + s^2 g_6)^2\right] = s^8 \delta'\,,
\end{aligned} \tag{2.2}$$

where f_3, f_4, q_2, g_5, g_6 are sections of certain line bundles over B: $f_3 \in H^0(B; \mathcal{T} \otimes \mathcal{L}^{\otimes 4})$, $f_4 \in H^0(B; \mathcal{L}^{\otimes 4})$, $q_2 \in H^0(B; \mathcal{T} \otimes \mathcal{L}^{\otimes 3})$, $g_5 \in H^0(B; \mathcal{T} \otimes \mathcal{L}^{\otimes 6})$ and $g_6 \in H^0(B; \mathcal{L}^{\otimes 6})$.

The fourfold described by (2.1) can be resolved to a nonsingular Calabi-Yau fourfold, in which the singularities of split IV^* type are replaced by rational curves whose intersection matrix reproduces the Dynkin diagram of E_6 (generically). No surprises are encountered during this resolution. One way to describe the resulting F-theory model is as a limit from three dimensions: first, compactify M-theory on the nonsingular Calabi-Yau fourfold, then consider the limit in which all fiber components introduced during the resolution of the Weierstrass model acquire zero area (leading to enhanced gauge symmetry), and finally, take the F-theory limit by sending the area of the elliptic fibers to zero, opening up a new effective dimension.

The transition (un-higgsing) to unbroken E_7 gauge group is described in F-theory terms by the condition $q_2 = 0$, which results in the singularity being enhanced to III^* fibers. Let us describe in some detail the geometry of this singular fourfold.

Under the condition $q_2 = 0$, the discriminant δ actually has a factor of s^9:

$$\delta = s^9\left[4(f_3 + sf_4)^3 + 27s(g_5 + sg_6)^2\right] = s^9 \delta'' \tag{2.3}$$

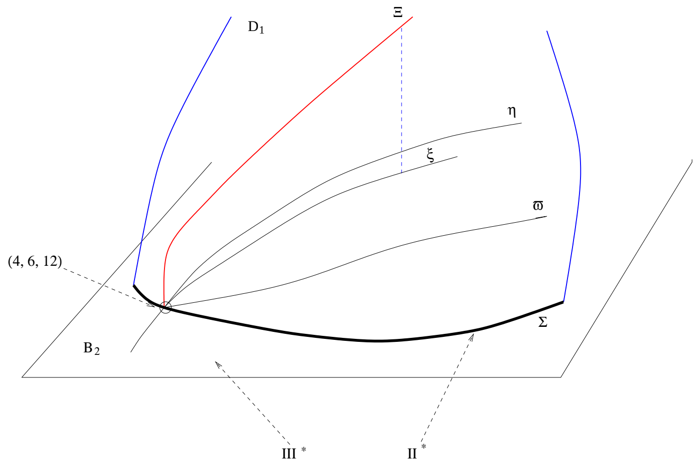

Figure 1: Geometry of the singular fourfold.

leading to III^* fibers. The component Δ'' of the discriminant is defined by the equation $\delta'' = 0$; this latter equation is a cubic equation in s whose discriminant with respect to s is given by:

$$\mathrm{discrim}(\delta'') = 2^4 \cdot 3^9 \cdot (f_4 g_5 - f_3 g_6)^3 \left(f_3^2 f_4 g_5 + g_5^3 - f_3^3 g_6\right). \tag{2.4}$$

Let Σ denote the locus $f_3 = 0$, therefore the matter curve is $\Sigma = 4c_1 - \Gamma$. Then, it is easy to see that the locus $g_5 = 0$ is precisely $\eta = 6c_1 - \Gamma$. Finally, by ξ and ω we denote the loci $f_4 g_5 - f_3 g_6 = 0$ and $f_3^2 f_4 g_5 + g_5^3 - f_3^3 g_6 = 0$ respectively. The singularity type is enhanced to II^* over the matter curve Σ, which is the intersection locus of S_0 with the nodal part of the discriminant, D_1. More interesting, the vanishing orders jump to $(4, 6, 12)$ over the intersection locus $\eta \cap \Sigma$. (In the familiar case of an elliptic surface, this would be the signal that the Weierstrass model was not minimal. However, in the present context there is no birational change which can be made which would reduce those orders of vanishing.) The set where the vanishing orders jump to $(4, 6, 12)$ is precisely the singularity set of the corresponding heterotic sheaf (which would otherwise be a bundle, were it not for the presence of this locus). There is a cusp curve Ξ inside D_1, which projects onto the curve ξ in B. The geometry of the singular fourfold is sketched in figure 1.

As explained in the introduction, resolution of this III^* locus results in the appearance of entire *complex surfaces* over the locus $\eta \cap \Sigma$ in B'. This phenomenon is most efficiently observed by performing a weighted blowup of the Weierstrass model, which we now proceed to describe.

2.2 The weighted blowup

The weighted blowup is performed by introducing an additional variable λ, and assigning weights as follows:

λ	s	x	y
-1	1	2	3

We now rewrite (2.1) as a homogeneous degree 5 equation in the variables λ, s, x, y as follows

$$y^2\lambda = x^3\lambda + s^3ux + s^5v\,. \tag{2.5}$$

This is the weighted blowup of (2.1). In the patch $\lambda \neq 0$, it is equivalent to (2.1), but when $\lambda = 0$, we get

$$s^3(ux + vs^2) = 0\,. \tag{2.6}$$

Note that over the point $u = v = 0$, this vanishes identically. Thus $u, v \in \mathbb{C}^2$ parametrise a family of hypersurfaces in $\mathbb{P}^{(1,2,3)}$ except at $u = v = 0$, when we obtain all of $\mathbb{P}^{(1,2,3)}$. This is precisely the complex surface mentioned before. Its occurance is a direct consequence of the $(4, 6, 12)$ vanishing orders of the discriminant along $u = v = 0$.

The weighted blowup is really only the first stage of a complete toric resolution of the III^* singularity, and thus does not introduce any extra Kähler classes beyond those already needed for the usual resolution of singularities. We present some further evidence for this lack of additional Kähler classes by studying an explicit toric example in the next subsection.[1]

2.3 Toric example

We now construct an explicit toric model of a Calabi-Yau fourfold elliptically fibered over base $B_3 = \mathbb{F}_{056}$ (i.e. $\Gamma = 5C_0 + 6f$, where C_0 and f are the classes in $B_2 = \mathbb{F}_0$), with a section $S_0 \simeq B_2 = \mathbb{F}_0$ of split IV^* singularities. This fourfold is dual to heterotic strings compactified on the $(3, 243)$ Calabi-Yau threefold which is elliptically fibered over $B_2 = \mathbb{F}_0$, with an $\mathrm{SU}(3) \times E_8$ bundle with $\eta_{\mathrm{SU}(3)} = 6c_1(B_2) - \Gamma = 7C_0 + 6f$, and $\eta_{E_8} = 6c_1(B_2) + \Gamma = 17C_0 + 18f$.

From index theorems and anomaly cancellation (see, for instance [13, 14]), we expect the following Hodge numbers, $h^{1,1} = 10$, $h^{3,1} - h^{2,1} = 9231$, $\chi = 55494$.

The Calabi-Yau fourfold may be constructed as a hypersurface in a toric variety following the prescription of [13, 14]. The dual polyhedron ∇, which encodes the divisors of the polyhedron has vertices:

$$\begin{array}{lllll} (-1,0,-6,2,3)\,, & (0,-1,-5,2,3)\,, & (0,0,0,-1,0)\,, & (0,0,0,0,-1)\,, & (0,1,0,2,3)\,, \\ (0,0,1,2,3)\,, & (1,0,0,2,3)\,, & (0,0,-2,1,1)\,, & (0,0,-1,0,0) & \end{array}$$

Standard toric methods [13, 14] give, for the Hodge numbers of the fourfold, $h^{1,1} = 10$, $h^{3,1} = 9231$, $h^{2,1} = 0$, $h^{2,2} = 37008$, $\chi = 55494$, in agreement with expectations. Note that the Euler characteristic is not divisible by 24, indicating that the model has a background G-flux turned on. Moreover, it is possible to find a triangulation of the polyhedron consistent with its elliptic fibration structure, such that each of the top dimensional cones has unit volume, guaranteeing smoothness of the corresponding Calabi-Yau fourfold. We assert that this polyhedron gives the F-theory dual of the heterotic vacuum described above.

[1] It is also possible to perform an explicit local resolution using the technique of [26], which would be related by generalized flops to the resolution presented in this section.

We can now study the effect of un-higgsing the unbroken E_6 gauge group to E_7. The heterotic bundle is now $\mathrm{SU}(2) \times E_8$, with $\eta_{\mathrm{SU}(2)} = 6c_1(B_2) - \Gamma = 7C_0 + 6f$, and η_{E_8} unchanged. Index theorems and anomaly cancellation predict, for the dual Calabi-Yau fourfold, Hodge numbers $h^{1,1} = 11$, $h^{3,1} - h^{2,1} = 9221$, $\chi = 55440$.

The dual polyhedron ∇ describing the Calabi-Yau fourfold has vertices

$$(-1,0,-6,2,3)\,,\quad (0,-1,-5,2,3)\,,\quad (0,0,0,-1,0)\,,\quad (0,0,0,0,-1)\,,\quad (0,1,0,2,3)\,,$$
$$(0,0,-1,0,1)\,,\quad (0,0,1,2,3)\,,\quad (1,0,0,2,3)$$

The fourfold has the following Hodge numbers, in accordance with our expectations: $h^{1,1} = 11$, $h^{3,1} = 9221$, $h^{2,1} = 0$, $h^{2,2} = 36972$, $\chi = 55440$. Once again, it is possible to find a triangulation of the of polyhedron consistent with its elliptic fibration structure, such that each of the top dimensional cones has unit volume, guaranteeing smoothness of the corresponding Calabi-Yau fourfold.

It should be emphasized here that no extra Kähler classes other than the ones corresponding to the resolution of the III^* locus are present in the fourfold. Since we expect, on general grounds, that the resolution of the singularity yields an entire complex surface over specific points in B_3, we conclude that *the appearance of the complex surface in the resolution of the III^* locus does not introduce any extra Kähler classes.*

The corresponding F-theory model has some features whose physical effects are difficult to explain in detail. We begin as before in three dimensions, with M-theory compactified on the nonsingular Calabi-Yau fourfold. When we allow the rational curves in the fibers to shrink to zero area, again we get enhanced gauge symmetry, but this time there are surfaces shrinking to points as well as curves shrinking to points. Wrapping the M-theory fivebrane on such surfaces suggests that the spectrum should contain light strings, while wrapping the M-theory membrane on curves within such surfaces would produce a tower of light particle states. All of these states are presumably present as well in the F-theory limit.

2.4 Comparison with codimension-two

It is worthwhile making a comparison between the geometry of these codimension-three singularities, and the analogous phenomenon in codimension-two. In the latter case, the F-theory interpretation of a small instanton singularity is that the total space of the elliptic fibration has acquired a singularity which can be resolved by a combination of blowing up the base of the fibration and blowing up the total space [6]. (It is the blowup of the base which leads to an additional branch of the moduli space.) In the codimension-three case, however, blowing up the base is not possible, because it destroys the Calabi-Yau condition.

To see this, consider first a simple model of the codimension-two phenomenon, represented by the Weierstrass equation

$$y^2 = x^3 + s^4 x + s^5 t\,. \tag{2.7}$$

One of the coordinate charts when blowing up the base is $s_1 = s$, $t_1 = t/s$; in that chart, the Weierstrass equation becomes

$$y^2 = x^3 + s_1^4 x + s_1^6 t_1\,. \tag{2.8}$$

To make this new Weierstrass equation minimal, we must also change coordinates in x and y, using $x_1 = x/s^2$, $y_1 = y/s^3$. Our final Weierstrass equation is then

$$y_1^2 = x_1^3 + x_1 + t_1. \tag{2.9}$$

(This two-step change of variables corresponds to the two-step geometric process of a blowup and a flop which was used in [6] to describe this transition.)

According to the Poincaré residue construction, the holomorphic three-form was originally represented by

$$\frac{dx \wedge ds \wedge dt}{2y}, \tag{2.10}$$

where $2y$ represents the partial derivative of (2.7) with respect to the variable y (which is not present in the numerator). In the new coordinate system (the minimal model of the blowup) this becomes

$$\frac{(s_1^2 dx_1) \wedge ds_1 \wedge (s_1 dt_1)}{2s_1^3 y_1} = \frac{dx_1 \wedge ds_1 \wedge dt_1}{2y_1}. \tag{2.11}$$

Since this latter is the Poincaré residue representation of a holomorphic three-form for the blown up threefold (2.9), our original three-form has acquired neither a zero nor a pole during this process. Thus, both threefolds can be Calabi-Yau and there is a transition between them. (Note that even though we only made the computation in a single coordinate chart, the order of zero or pole of the holomorphic three-form would be the same in *any* coordinate chart, so this is actually a complete argument.)

By contrast, let us make a similar computation for a fourfold, starting from the Weierstrass equation

$$y^2 = x^3 + s^3 ux + s^5 v\,. \tag{2.12}$$

We can represent one of the coordinate charts of the blowup by $s_1 = s$, $u_1 = u/s$, $v_1 = v/s$; in that chart, the Weierstrass equation becomes

$$y^2 = x^3 + s_1^4 u_1 x + s_1^6 v_1\,. \tag{2.13}$$

We again get a non-minimal Weierstrass model, which can be made minimal by the further coordinate change $x_1 = x/s^2$, $y_1 = y/s^3$. Our final Weierstrass equation is then

$$y_1^2 = x_1^3 + u_1 x_1 + v_1\,. \tag{2.14}$$

The holomorphic four-form was originally represented by

$$\frac{dx \wedge ds \wedge du \wedge dv}{2y\,,} \tag{2.15}$$

and in the minimal model of the blowup this becomes

$$\frac{(s_1^2 dx_1) \wedge ds_1 \wedge (s_1 du_1) \wedge (s_1 dv_1)}{2s_1^3 y_1} = \frac{s_1 dx_1 \wedge ds_1 \wedge du_1 \wedge dv_1}{2y_1}. \tag{2.16}$$

Since this is s_1 times the Poincaré residue representation of a holomorphic four-form for the blown up fourfold (2.14), our original four-form has acquired a zero along the exceptional divisor $s_1 = 0$. Thus, at most one of these two fourfolds can be Calabi-Yau (i.e. have a *non-vanishing* holomorphic four-form), and there is no physical transition between them.

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3. Singular bundles and transitions

3.1 Spectral data

We begin with a short review of the spectral cover approach to bundles on elliptic fibrations [20]. Let $\pi : Z \to B$ be a smooth elliptic Calabi-Yau variety with a section $\sigma : B \to Z$. As usual, we assume that Z is moreover a cubic hypersurface in $\mathbb{P}\left(\mathcal{O}_B \oplus \mathcal{L}^2 \oplus \mathcal{L}^3\right)$, where $\mathcal{L} = \mathcal{O}_B(-K_B)$ is the anticanonical line bundle of the base.

According to [20], the moduli space of rank n semistable bundles with trivial determinant on a smooth elliptic curve E is isomorphic to the linear system $|np_0| \simeq \mathbb{P}^{n-1}$, where p_0 is the origin of E. This construction works for families of elliptic curves as well, if the singular fibers are either nodal or cuspidal curves. For the Weierstrass model $Z \to B$ introduced before, this yields a relative coarse moduli space $\mathcal{P}_{n-1} \to B$ which is isomorphic to the relative projective space $\mathbb{P}\left(\mathcal{O}_N \oplus \mathcal{L}^{-2} \oplus \cdots \oplus \mathcal{L}^{-n}\right)$. If $V \to Z$ is a rank n bundle whose restriction to every fiber E_b is semistable and regular with trivial determinant, then V determines a section $A : B \to \mathcal{P}_{n-1}$. Such a section A is uniquely given by a line bundle $\mathcal{M}$ over B, and sections $a_i \in H^0\left(B, \mathcal{M} \otimes \mathcal{L}^{-i}\right)$, $i = 0, 2, \ldots, n$.

The converse is not true, i.e. a section A does not uniquely determine a bundle V. Friedman, Morgan, and Witten [20] construct certain basic bundles $V_{A,a}$ associated to a section A together with an integer $a \in \mathbb{Z}$. The construction is rather involved and it will not be reviewed here in detail. After some work, it can be shown that it is equivalent to the standard spectral cover construction [20]. Namely, the section A determines a spectral cover $C_A \subset Z$ which belongs to the linear system $|n\sigma + \pi^*\alpha|$, where $\alpha = c_1(\mathcal{M})$. In order to construct $V_{A,a}$, let us consider the following diagram [20]

$$\begin{tikzcd} T_A \arrow[r, "\nu_A"] \arrow[d, "\rho_A"'] & Z \arrow[d, "\pi"] \\ C_A \arrow[r, "g_A"] & B. \end{tikzcd}$$

Note that $T_A \to C_A$ is an elliptic fibration with a section $\Sigma_A = \nu_A^*\sigma$. There are certain natural Weil divisor classes on T_A: the diagonal Δ obtained by restriction from $Z \times_B Z$ and the class $\rho_A^*(F)$, where $F = C_A \cdot \sigma$.[2] Then we have

$$V_{A,a} = (\nu_A)_* \mathcal{O}_{T_A} \left(\Delta - \Sigma_A - a\rho_A^* F\right). \tag{3.1}$$

At this point, it may be helpful to compare the present approach to the more familiar construction of [23]. Let $\mathcal{P} \to Z \times_B Z$ denote the universal Poincaré line bundle and let $\mathcal{N}$ denote a line bundle over the spectral cover C_A. Note that

$$\mathcal{P} = \mathcal{O}_{Z \times_B Z} \left(\Delta - \sigma_1 - \sigma_2\right) \otimes \mathcal{L}^{-1}. \tag{3.2}$$

Then we construct a bundle V associated to the pair $(C_A, \mathcal{N})$

$$V = \nu_{A*} \left(\rho_A^* \mathcal{N} \otimes \mathcal{P}_T\right). \tag{3.3}$$

[2] Δ is not a Cartier divisor near the singular locus of T_A.

We can establish a direct relation between (3.1) and (3.3) by noting that $\sigma_1|_{T_A} = \Sigma_A$ and $\sigma_2|_{T_A} = F$. Therefore we must have

$$\mathcal{P}|_{T_A} = \mathcal{O}_{T_A}(\Delta - \Sigma_A - F) \otimes \mathcal{L}^{-1}, \qquad \mathcal{N} = \mathcal{O}_{C_A}(-(a-1)F) \otimes \mathcal{L}\,. \tag{3.4}$$

We can construct more general bundles by twisting by an arbitrary line bundle $\mathcal{S}$ pulled back from B.

The Chern classes of the bundles $V_{A,a}$ are given by [20]

$$\mathrm{ch}\,(V_{A,a}) = e^{-\alpha}\frac{1-e^{(a+n)L}}{1-e^{L}} - \frac{1-e^{aL}}{1-e^{L}} + e^{-\sigma}(1-e^{-\alpha})\,, \tag{3.5}$$

where $L = c_1(\mathcal{L})$. From now on, we will supress the indices A, a for simplicity. The bundles will be denoted by V_n in order to emphasize their rank.

3.2 Reducible spectral cover

Generically, the spectral cover C is smooth and irreducible, at least if the line bundle $\mathcal{M}$ is sufficiently ample on B. For physics reasons, we have nevertheless to understand the bundles V associated to reducible spectral covers. Following [20, section 5.7] we consider here the case when the spectral cover C splits as a sum $C' + \sigma$, but is otherwise generic. As explained in detail in [20], such a degeneration of C corresponds to a section $A : B \to \mathcal{P}_{n-1}$ which lies in the hyperplane $\mathcal{P}_{n-2} \subset \mathcal{P}_{n-1}$. C' corresponds to a rank $n-1$ bundle V_{n-1} and we have the following elementary modification

$$0 \to V_n \to V_{n-1} \oplus \pi^*\mathcal{L}^a \to \pi^*\mathcal{L}^a|_{\pi^*F} \to 0\,. \tag{3.6}$$

Here we view $F = C \cdot \sigma$ as a divisor on $\sigma \simeq B$, hence π^*F is the vertical divisor above F. In terms of divisor classes on B,

$$F = \alpha - (n-1)L\,. \tag{3.7}$$

Note that by construction, there is a surjective morphism $V_n \to V_{n-1}$ whose kernel is isomorphic to $\pi^*\mathcal{L}^a(-F)$. Therefore we have an exact sequence

$$0 \to \pi^*\mathcal{L}^a(-F) \to V_n \to V_{n-1} \to 0\,. \tag{3.8}$$

The rank changing transition described above is valid for n sufficiently large such that A is a regular section of both $\mathcal{P}_{n-2}$ and $\mathcal{P}_{n-1}$. If n is small, i.e. $n \leq 3$, this is no longer true and the above picture must change. Understanding the new picture will be our main goal in the remaining part of this section.

3.3 Singular bundles and reflexive sheaves

For small values of n (such as $n = 3$ in the present case) we cannot have regular sections $A : B \to \mathcal{P}_{n-2}$. Recall that a section $A : B \to \mathcal{P}_1$ would be defined by a line bundle $\mathcal{M}$ on B together with two sections $a_0 \in H^0(B, \mathcal{M})$, $a_2 \in H^0(B, \mathcal{M} \otimes \mathcal{L}^{-2})$. Since the base B is two dimensional, the sections a_0, a_2 will have generically common zeroes at finitely many points $b_1, \ldots, b_k$ on B. This means that the section A is defined only on the complement of $\{b_i\}$ in B. The closure of the image of A in $\mathcal{P}_1$ is isomorphic to the blowup $\widetilde{B}$ of B at the points $\{b_i\}$. The exceptional curves on $\widetilde{B}$ coincide with the $\mathbb{P}^1$ fibers projecting to the points B_i. We will call such a section A a quasisection.

Given a quasisection A, and an integer $a \in \mathbb{Z}$, we can construct a rank 2 bundle V_2 over the open subset $Z \setminus \cup E_i$, where $E_i = \pi^{-1}(b_i)$. However, it is not clear if V_2 can be extended as a bundle over the threefold Z. In general, even if an extension exists, we expect some kind of singular behavior along the elliptic fibers E_i.

This problem has been addressed in [20] where it has been shown that V_2 can always be extended as a sheaf, and the singular behavior depends on the parity of a. We now recall the main construction and the central results. The idea is, roughly, to work over the blown up base $\widetilde{B}$ where we can apply the standard construction and then push-forward to B. Let us consider the diagram

$$\begin{tikzcd} \widetilde{\mathcal{P}}_1 \arrow[r] \arrow[d] & \mathcal{P}_1 \arrow[d] \\ \widetilde{B} \arrow[r] & B. \end{tikzcd}$$

The pullback of the rational section A to $\widetilde{\mathcal{P}}_1$ is a regular section $\widetilde{A} : \widetilde{B} \to \widetilde{\mathcal{P}}_1$. Therefore we can construct a rank 2 bundle $\widetilde{V}_2$ over $\widetilde{Z} = \widetilde{B} \times_B Z$. Note that $\widetilde{Z}$ is the blowup of Z along the elliptic curves E_i, with exceptional divisors $D_i = E_i \times \mathbb{P}^1$. Let $q : \widetilde{Z} \to Z$ denote the blowup map. Then, the direct image $q_* \widetilde{V}_2$ defines in principle an extension of V_2. However, in order to keep the singular behavior along the elliptic fibers E_i under control, we have to perform a more elaborate construction. The result depends on the behavior of $\widetilde{V}_2$ along the exceptional divisors of D_i, which in turn depends on the parity of a.

If a is even, the bundle V_2 can be extended over Z as a bundle whose restriction to the fibers E_i is unstable. This is a rather mild singular behavior and we will not discuss this case further in this paper.

If a is odd, the bundle V_2 can be extended as a reflexive sheaf over Z, whose construction will be reviewed here in some detail. Note that we will denote the extension also by V_2, the meaning being clear from the context. In order to avoid unnecessary complications, we will also fix $a = 1$ and work near a fixed fiber E_i, dropping the index i.

The restriction of $\widetilde{V}_2$ to the exceptional divisor $D = E \times \mathbb{P}^1$ is [20]

$$\widetilde{V}_2|_{\mathbb{P}^1 \times p} = \begin{cases} \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)\,, & \text{if } p \neq p_0 \\ \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)\,, & \text{if } p = p_0\,. \end{cases} \tag{3.9}$$

where p_0 is the origin of E. In order for the direct image to be well behaved, we would like $\widetilde{V}_2$ to be trivial along the exceptional fibers $\mathbb{P}^1$. This can be realized by first twisting by $\mathcal{O}_{\widetilde{Z}}(-D)$ and then performing an elementary modification along the curve $\mathbb{P}^1 \times p_0$. However, an elementary modification in codimension two does not result in a locally free sheaf. Therefore, we need to perform a second blowup along $\mathbb{P}^1 \times p_0$. Let $g : Z_1 \to \widetilde{Z}$ denote the blowup map with exceptional divisor $D_1 \simeq \mathbb{P}^1\left(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)\right)$. Let D' denote the proper transform of D to Z_1, that is, $g^*D = D' + D_1$; D' and D_1 intersect transversely along the negative section of D_1. Note that D' is a rationally ruled surface over E. Finally we can contract D' along its fibers obtaining a threefold Z_2 which is the blowup of Z at the point $p_0 \in E$. The exceptional divisor $D_2 \simeq \mathbb{P}^2$ is obtained by contracting the negative section on $D_1 \simeq \mathbb{F}_1$. This sequence of birational transformations can be summarized in the following diagram

$$\begin{tikzcd} Z_1 \arrow[r, "g"] \arrow[d, "p"'] & \widetilde{Z} \arrow[d, "q"] \\ Z_2 \arrow[r, "f"] & Z. \end{tikzcd}$$

Now let us describe the bundle construction [20]. We first pull back $\widetilde{V}_2(-D)$ to Z_1 and perform an elementary modification along D_1

$$0 \to V_2' \to g^*\widetilde{V}_2(-D) \to j_* \left(\mathcal{O}_{D_1}(-f)\right) \to 0\,, \tag{3.10}$$

where $j : D_1 \to Z_1$ is the inclusion and f is the fiber class of $D_1 \simeq \mathbb{F}_1$. It turns out that V_2' is locally free and uniformly trivial along the fibers of D'. Then, it follows that p_*V_2' is also locally free, and with some work it can be shown that its restriction to the exceptional divisor $D_2 = \mathbb{P}^2$ is isomorphic to $T_{\mathbb{P}^2}(-1)$.

The final step is to take the direct image $V_2 = f_*p_*V_2'$ on Z. Given the above behavior along D_2, it can be shown using the formal functions theorem [20] that V_2 is a reflexive sheaf on Z. This concludes the extension of the rank two bundle defined over $Z \setminus \cup E_i$ to Z.

3.4 The transition

In this subsection we generalize the rank changing transition discussed in section 3.2 to $n = 3$. In order to fix ideas, recall that we start with a smooth rank three bundle V_3 corresponding to a regular section of $\mathcal{P}_2 \to B$. Our goal is to understand the bundle degeneration associated to a deformation of this regular section to a quasisection $A : B \to \mathcal{P}_1 \subset \mathcal{P}_2$. For simplicity, we will assume that there is a single point $b \in B$ where A fails to be a section. The generalization to a finite collection of points $\{b_1, \ldots, b_k\}$ is straightforward.

The main idea is to work again on the blown up space $\widetilde{Z}$ where we can apply the standard theory, and then push forward to Z. However, there are extra complications arising from the necessity of a second blowup, as discussed above. The quasisection A pulls back to a section $\widetilde{A}$ of $\widetilde{\mathcal{P}}_2$ over $\widetilde{B}$ whose image lies in $\widetilde{\mathcal{P}}_1$, but is otherwise generic. Therefore we can construct a rank three bundle as in section 3.2

$$0 \to \widetilde{V}_3 \to \widetilde{V}_2 \oplus \mathcal{L}^a \to \mathcal{L}^a|_{\widetilde{F}} \to 0\,, \tag{3.11}$$

where

$$\widetilde{F} = \widetilde{\alpha} - 2L = \alpha - D - 2L\,. \tag{3.12}$$

Note that $\widetilde{F}, \alpha, L$ should be understood now as divisor classes on $\widetilde{Z}$ obtained by pull-back from $\widetilde{B}$. We have dropped the explicit notation $\widetilde{\pi}^*$ for simplicity. In the following, it will be more convenient to use the alternative presentation of $\widetilde{V}_3$

$$0 \to \mathcal{L}^a(-\widetilde{F}) \to \widetilde{V}_3 \to \widetilde{V}_2 \to 0\,. \tag{3.13}$$

Next we proceed step by step, by analogy with the construction of the reflexive sheaf V_2 in the previous section. Twisting by $\mathcal{O}_{\widetilde{Z}}(-D)$, and taking into account (3.12), we obtain the following exact sequence

$$0 \to \mathcal{L}^{a+2}(-\alpha) \to \widetilde{V}_3(-D) \to \widetilde{V}_2(-D) \to 0\,. \tag{3.14}$$

We have

$$\begin{aligned} \widetilde{V}_2(-D)|_{\mathbb{P}^1 \times p_0} &= \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \\ \widetilde{V}_3(-D)|_{\mathbb{P}^1 \times p_0} &= \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1} \end{aligned} \tag{3.15}$$

since (3.14) splits over $\mathbb{P}^1 \times p_0$. Moreover, since $\mathcal{L}^{a+2}(-\alpha)$ is trivial along the fibers of $D \to E$, it follows from (3.9) and (3.14) that

$$\widetilde{V}_3(-D)|_{\mathbb{P}^1 \times p} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \tag{3.16}$$

for $p \neq p_0$. Following the steps outlined above, we pull back (3.14) to Z_1, and perform an elementary modification along D_1. Note that (3.15) implies

$$\begin{aligned} g^*\widetilde{V}_2(-D)|_{D_1} &= \mathcal{O}_{D_1}(f) \oplus \mathcal{O}_{D_1}(-f) \\ g^*\widetilde{V}_3(-D)|_{D_1} &= \mathcal{O}_{D_1}(f) \oplus \mathcal{O}_{D_1}(-f) \oplus \mathcal{O}_{D_1}\,, \end{aligned} \tag{3.17}$$

therefore we can perform elementary modifications of $g^*\widetilde{V}_2(-D), g^*\widetilde{V}_3(-D)$ along D_1. We obtain the following commutative exact diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{L}^{a+2}(-\alpha) \arrow[r] \arrow[d] & V_3' \arrow[r] \arrow[d] & V_2' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{L}^{a+2}(-\alpha) \arrow[r] \arrow[d] & g^*\widetilde{V}_3(-D) \arrow[r] \arrow[d] & g^*\widetilde{V}_2(-D) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & j_*\mathcal{O}_{D_1}(-f) \arrow[r] \arrow[d] & j_*\mathcal{O}_{D_1}(-f) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Given the above diagram, the results of [20] show that $f_*p_*V_3'$ is a rank 3 reflexive sheaf of Z. One might think that this is the solution to our problem, namely that $f_*p_*V_3'$ can then be deformed to a smooth rank 3 bundle corresponding to a generic section of $\mathcal{P}_2 \to B$. It turns out however that this is not true since the sheaf $f_*p_*V_3'$ has the *wrong topological invariants.* This fairly elaborate computation is postponed to appendix.

In order to obtain a sheaf with the correct topological invariants, we have to perform a further elementary modification on ${V'}_3$. By restricting the top row of the above diagram to D_1, we obtain the following exact sequence

$$0 \to \mathcal{O}_{D_1} \to {V'}_3|_{D_1} \to {V'}_2|_{D_1} \to 0, \tag{3.18}$$

where $V'_2|_{D_1} \simeq p^*T_{\mathbb{P}^2}(-1)$, [20, claim 2, p. 87]. It can be checked that $\mathrm{Ext}^1(p^*T_{\mathbb{P}^2}(-1), \mathcal{O}_{D_1}) = 0$, therefore $V'_3|_{D_1} \simeq \mathcal{O}_{D_1} \oplus p^*T_{\mathbb{P}^2}(-1)$. Now we can perform the elementary modification

$$0 \to V''_3 \to V'_3 \to j_*\mathcal{O}_{D_1} \to 0\,. \tag{3.19}$$

We claim that $f_*p_*V''_3$ represents the solution to our problem.

In the following we will examine the properties of $f_*p_*V''_3$ and derive a simple relation between $f_*p_*V''_3$ and $f_*p_*V'_2$. Along the way we will prove that all the higher direct images of the form $R^ip_*V''_3$, $R^if_*(p_*V''_3)$ vanish. Combined with the Chern class computations of appendix, this shows that $f_*p_*V''_3$ has the right topological invariants.

Recall that the map $p : Z_1 \to Z_2$ contracts the fibers of D', which is a $\mathbb{P}^1$-fibration over E. Note that V''_3 is trivial along all fibers except possibly the fiber projecting to $p_0 \in E$. This follows from (3.16). The fiber over p_0 coincides with the negative section l of D_1. By construction, we have an exact sequence

$$0 \to \mathcal{O}_{D_1}(l+f) \to V''_3|_{D_1} \to p^*T_{\mathbb{P}^2}(-1) \to 0 \tag{3.20}$$

which restricts to

$$0 \to \mathcal{O}_{\mathbb{P}^1} \to V''_3|_l \to \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \to 0\,. \tag{3.21}$$

This shows that V''_3 is also trivial along l. Therefore, we can contract the fibers of D', obtaining a vector bundle $p_*V''_3$ on Z_2. Moreover, using the base change theorem, it can be shown that $R^ip_*V''_3 = 0$ for all $i \geq 1$, as claimed before. Then, the exact sequence (3.19) induces a similar exact sequence on Z_2

$$0 \to p_*V''_3 \to p_*V'_3 \to j_*\mathcal{O}_{D_2} \to 0 \tag{3.22}$$

which shows that $p_*V''_3$ is an elementary modification of $p_*V'_3$. By computing extensions, it can be shown that (3.20) must be split, $V''_3|_{D_1} = \mathcal{O}_{D_1}(l+f) \oplus p^*T_{\mathbb{P}^2}(-1)$. After contraction, this yields $p_*V''_3|_{D_2} = \mathcal{O}_{\mathbb{P}^2}(1) \oplus T_{\mathbb{P}^2}(-1)$, which implies $R^if_*(p_*V''_3) = 0$ for all $i \geq 1$. Therefore we have an exact sequence

$$0 \to f_*p_*V''_3 \to f_*p_*V'_3 \to \mathcal{O}_{p_0} \to 0\,. \tag{3.23}$$

In order to derive a direct relation with $f_*p_*V'_2$, note that we have another short exact sequence on Z

$$0 \to \mathcal{L}^{a+2}(-\alpha) \to f_*p_*V'_3 \to f_*p_*V'_2 \to 0\,. \tag{3.24}$$

Combining (3.23) and (3.24) we obtain

$$0 \to \mathcal{J}_{p_0} \otimes \mathcal{L}^{a+2}(-\alpha) \to f_*p_*V''_3 \to f_*p_*V'_2 \to 0\,, \tag{3.25}$$

which is the promised relation.

Moreover, (3.23) shows that the local behavior of the sheaf $f_*p_*V''_3$ near p_0 is of the form $V_{2p_0} \oplus \mathcal{J}_{p_0}$ where V_{2p_0} is a reflexive rank two $\mathcal{O}_{Z,p_0}$-module. The generalization of (3.25) to a finite collection of singular points $\{b_i\}$ is straightforward–we simply replace $\mathcal{J}_{p_0}$ by $\mathcal{J}_{p_{01}+\cdots+p_{0k}}$.

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3.5 The heterotic transition

The bundles $V_{A,a}$ studied above are perhaps the most natural in the spectral cover construction, but they are not quite suitable for physics. In order to define a consistent heterotic background we need bundles with $c_1(V) = 0$ and $c_2(V) = c_2(X)$. The second condition can be relaxed if we allow heterotic fivebranes wrapping holomorphic curves in the Calabi-Yau threefold. In order to satisfy the first condition, one can try to tensor the bundles $V_{A,a}$ by a line bundle $\mathcal{N}_0$ pulled back from B. By expanding (3.5), we find

$$c_1\left(V_{A,a} \otimes \mathcal{N}_0\right) = \left(an + \frac{n(n-1)}{2}\right) L - (a+n-1)\alpha + nc_1(\mathcal{N}_0)\,. \tag{3.26}$$

Notice that if n is odd, we can find a generic solution to $c_1\left(V_{A,a} \otimes \mathcal{N}_0\right) = 0$ by taking $a = 1$ and

$$c_1(\mathcal{N}_0) = \alpha - \frac{n+1}{2} L\,. \tag{3.27}$$

Since our aim is to describe $E_6 \to E_7$ transition, we start with $n = 3,\ a = 1$, therefore the heterotic data is specified by a rank 3 bundle $U_3 = V_3 \otimes \pi^*\mathcal{O}_B(\alpha - 2L)$. After transition we have a rank two reflexive sheaf V_2 related to V_3 by the exact sequence (3.25). Then it is harder to find generic solutions to $c_1(V_2 \otimes \mathcal{N}_0) = 0$. In fact, it is clear that such a bundle exists if and only if

$$L \equiv 0 \bmod 2\,. \tag{3.28}$$

This condition is rather restrictive, but it is satisfied if the base of the elliptic fibration is for example $B = \mathbb{F}_0$ or $B = \mathbb{F}_2$. If (3.28) is not satisfied, we will have to understand more complicated transitions which change a as well as n. This is an interesting problem, but we will not address it here.

Let us assume for now that $L = 2L'$, i.e. the anticanonical line bundle $\mathcal{L}$ has a square root $\mathcal{L}'$. Then the heterotic data after the transition will be specified by the reflexive sheaf $U_2 = V_2 \otimes \mathcal{O}_B\left(\alpha - 3L'\right)$ The exact sequence (3.25) implies

$$0 \to \mathcal{J}_{p_{01}+\cdots+p_{0k}} \otimes \mathcal{L} \to U_3 \to U_2 \otimes {\mathcal{L}'}^{-1} \to 0\,. \tag{3.29}$$

To conclude, it might be helpful to express the bundle data in the language of [23]. Using the relations, (3.4) a straightforward computation shows that the solution (3.27) is equivalent to the choice

$$\lambda = \frac{1}{2} \tag{3.30}$$

in the notation of [23]. The number λ determines the third Chern class of the heterotic bundle U_n. More precisely, for a smooth generic bundle we have [27]

$$c_3(U_n) = 2\lambda\alpha(\alpha - nL)\,. \tag{3.31}$$

This formula is not valid for the rank two reflexive sheaf U_2 constructed above. As detailed in appendix, the third Chern class of U_2 is

$$c_3(U_2) = 2(1+\lambda)\alpha(\alpha - 2L)\,. \tag{3.32}$$

Note that $\alpha(\alpha - 2L)$ is precisely the number of points on Z where U_2 is not locally free, i.e. the number of blowups required in the previous construction. This shift in the third Chern class is related to the presence of a nonperturbative sector after the transition. According to the evidence known so far [22], the reflexive sheaf U_2 defines a nonsingular $(0,2)$ CFT. However, the complete set of data specifying the heterotic vacuum also includes a nonperturbative sector consisting of massless excitations localized along certain three dimensional space-time defects. These defects are associated to the ideal sheaves $\mathcal{J}_{p_{0i}}$ and contribute $-2\alpha(\alpha-2L)$ to the third Chern class. This is quite similar to the small instanton transition in six dimensions, where the fivebranes contribute to the second Chern class of the gauge bundle. The main difference is that in the present case there is no Coulomb branch; the defects cannot leave the Hořava-Witten boundary.

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A. Chern classes

Here we compute the Chern character of the K theoretic direct images $f_!p_!V'_3$, $f_!p_!V''_3$ using the Grothendieck-Riemann-Roch (GRR) theorem. Recall that we have a basic diagram

$$\begin{tikzcd} Z_1 \arrow[r, "g"] \arrow[d, "p"'] & \widetilde{Z} \arrow[d, "q"] \\ Z_2 \arrow[r, "f"] & Z\,. \end{tikzcd}$$

In order to apply the GRR theorem, we need to know the Chern classes of the manifolds $\widetilde{Z}, Z_1, Z_2$ in terms of those of Z and the classes of the exceptional divisors. The general formula [28] is as follows.

Let X be a projective variety and $Y \subset X$ be a smooth subvariety with normal bundle $\mathcal{E}$. Let $f : X' \to X$ be the blowup of X along Y and let $j : D \hookrightarrow X'$ be the exceptional divisor. Let $g : D = \mathbb{P}\mathcal{E} \to Y$ denote the standard projection. We define a class $v \in H^2(D)$ by

$$v = j^* j_*(\mathbf{1}_D)\,. \tag{A.1}$$

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Then we have the formula

$$f^*\mathrm{ch}(X) - \mathrm{ch}(X') = j_*\left[(g^*\mathrm{ch}\mathcal{E} - e^v)\left(\frac{1-e^{-v}}{v}\right)\right]. \tag{A.2}$$

Now let us summarize the notations and conventions for the above diagram.

$q : \widetilde{Z} \to Z$	The blowup of Z along a smooth elliptic fiber of Z.
D	The exceptional divisor of $q : \widetilde{Z} \to Z$.
E	The proper transform of the elliptic fiber; $D = E \times \mathbb{P}^1$, $N_{D/\widetilde{Z}} = \mathcal{O}_D(-E)$.
$g : Z_1 \to \widetilde{Z}$	The blowup of $\widetilde{Z}$ along the exceptional fiber $\mathbb{P}^1 \times p_0$, where p_0 is the origin of E.
D_1	The exceptional divisor of g; $D_1 \simeq \mathbb{F}_1$.
l	The proper transform of $\mathbb{P}^1 \times p_0$; l is the negative section of D_1.
f	The fiber class of D_1.
$h = l + f$	The positive section of $D_1 = \mathbb{F}_1$; $N_{D_1/Z_1} = \mathcal{O}_{D_1}(-h)$.
D'	The proper transform of D; $g^*D = D' + D_1$.
E'	The proper transform of E, i.e. $g^*E = E' + f$; $N_{D'/Z_1} = \mathcal{O}_{D'}(-E' - l)$.
$p : Z_1 \to Z_2$	Contraction along the $\mathbb{P}^1$ fibers of D'.
$f : Z_2 \to Z$	The blowup of Z at the point $p_0 = E \cap \sigma$.
$D_2 = \mathbb{P}^2$	The exceptional divisor of $f : Z_2 \to Z$; $D_2 = p_* D_1$.

Using (A.2), we compute

$$\begin{aligned}
\mathrm{Td}(\widetilde{Z}) &= 1 - \frac{D}{2} + \mathrm{Td}_2(Z) \\
\mathrm{Td}(Z_2) &= 1 - D_2 + \mathrm{Td}_2(Z) - \frac{h}{3} \\
\mathrm{Td}(Z_1) &= 1 - \left(D_1 + \frac{D'}{2}\right) + \mathrm{Td}_2(Z) - \frac{h-l}{3} \\
&= 1 + \mathrm{Td}_1(Z_2) - \frac{D'}{2} + \mathrm{Td}_2(Z_2) + \frac{l}{3}
\end{aligned} \tag{A.3}$$

Let x be a K theory class in $K(Z_1)$. Then we have

$$\begin{aligned}
\mathrm{ch}_0(p_!x) &= \mathrm{ch}_0(x) \\
\mathrm{ch}_1(p_!x) &= p_*\mathrm{ch}_1(x) \\
\mathrm{ch}_2(p_!x) &= p_*\left[\mathrm{ch}_2(x) - \frac{D'}{2}\mathrm{ch}_1(x)\right] \\
\mathrm{ch}_3(p_!x) &= p_*\left[\mathrm{ch}_3(x) - \frac{D'}{2}\mathrm{ch}_2(x) - \frac{l}{6}\mathrm{ch}_1(x)\right]
\end{aligned} \tag{A.4}$$

Now let y be a K theory class in $K(Z_2)$. Then

$$\begin{aligned}
\mathrm{ch}_0(f_!y) &= \mathrm{ch}_0(y) \\
\mathrm{ch}_1(f_!y) &= f_*\mathrm{ch}_1(y) \\
\mathrm{ch}_2(f_!y) &= f_*\left[\mathrm{ch}_2(y) - D_2\mathrm{ch}_1(y)\right] \\
\mathrm{ch}_3(f_!y) &= f_*\left[\mathrm{ch}_3(y) - D_2\mathrm{ch}_2(y) - \frac{h}{3}\mathrm{ch}_1(y)\right]
\end{aligned} \tag{A.5}$$

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Given these formulae, one can compute the topological invariants of the K theory classes $f_! p_! V'_2$, $f_! p_! V'_3$, $f_! p_! V''_3$. The starting point is the spectral cover construction on $\widetilde{Z}$. In general, this produces a smooth rank n bundle with Chern character

$$\mathrm{ch}(\widetilde{V}_n) = e^{-\widetilde{\alpha}} \left(\frac{1 - e^{(a+n)L}}{1 - e^L} \right) - \frac{1 - e^{aL}}{1 - e^L} + e^{-\widetilde{\sigma}} (1 - e^{-\widetilde{\alpha}}) \,. \tag{A.6}$$

Recall that $\widetilde{\alpha}, \widetilde{\sigma}$ are the proper transforms of α, σ,

$$\widetilde{\alpha} = q^* \alpha - D \,, \qquad \widetilde{\sigma} = q^* \sigma \,. \tag{A.7}$$

The final result can be conveniently expressed in terms of the Chern characters of the generic smooth bundle (3.5). More precisely, let us define $\mathrm{ch}(n,a)$ by the formula

$$\mathrm{ch}(n,a) = e^{-\alpha} \frac{1 - e^{(a+n)L}}{1 - e^L} - \frac{1 - e^{aL}}{1 - e^L} + e^{-\sigma} (1 - e^{-\alpha}) \,. \tag{A.8}$$

Then we find (for $a = 1, k = 0$)

$$\begin{aligned} \mathrm{ch}(f_! p_! V'_2) &= \mathrm{ch}(2,a) + w_Z \\ \mathrm{ch}(f_! p_! V'_3) &= \mathrm{ch}(3,a) + w_Z \\ \mathrm{ch}(f_! p_! V''_3) &= \mathrm{ch}(3,a) \,, \end{aligned} \tag{A.9}$$

where w_Z is the fundamental class of Z. An useful intermediate result is

$$\mathrm{ch}\left(f_! p_! j_* \left(\mathcal{O}_{D_1}(-(m+1)f) \right) \right) = -m w_Z \,. \tag{A.10}$$

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